TUTORIAL NOTES FOR MATH4010

JUNHAO ZHANG

1. DUAL SPACE OF ℓ^p

Let us discuss the dual space of ℓ^p.

Example 1 ($(\ell^p)^* = \ell^q$)**.** Let $1 \le p < \infty$, for given $y \in \ell^q$, define $T_y : \ell^p \to \mathbb{R}$ by

$$T_y(x) = \sum_{i=1}^{\infty} x(i)y(i), \quad \forall x \in \ell^p,$$

Then $T : y \mapsto T_y$ is an isometric isomorphism of the Banach space ℓ^q onto the dual of ℓ^p, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We define the pairing for $x \in \ell^p$ and $y \in \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$\langle x, y\rangle := \sum_{i=1}^{\infty} x(i)y(i).$$

This expression is well-defined. Indeed, by the Hölder's inequality,

$$|\langle x, y\rangle| \le \sum_{i=1}^{\infty} |x(i)y(i)| \le \|x\|_p \|y\|_q < \infty.$$

Therefore for any fixed $y \in \ell^q$, we can define the a linear continuous functional

$$\begin{aligned} f_y : \ell^p &\to \mathbb{R}, \\ x &\mapsto \langle x, y\rangle. \end{aligned}$$

Hence $f_y \in (\ell^p)^*$. Furthermore, we define the map

$$\begin{aligned} T : \ell^q &\to (\ell^p)^*, \\ y &\mapsto f_y. \end{aligned}$$

We claim that T is an isomorphism, i.e. T is a linear bijection such that

$$\|T(y)\|_{(\ell^p)^*} = \|y\|_q. \quad \text{(isometric mapping)}$$

Since for arbitrary $x \in \ell^p$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, $y_1, y_2 \in \ell^q$,

$$f_{\lambda_1 y_1 + \lambda_2 y_2}(x) = \langle x, \lambda_1 y_1 + \lambda_2 y_2\rangle = \lambda_1\langle x, y_1\rangle + \lambda_2\langle x, y_2\rangle = \lambda_1 f_{y_1}(x) + \lambda_2 f_{y_2}(x),$$

which implies

$$T(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 T(y_1) + \lambda_2 T(y_2).$$

Therefore T is linear.

To prove T is injective, suppose $f_y = 0$, then for arbitrary $x \in \ell^p$,

$$\langle x, y\rangle = 0,$$

we take $x = e_i$, where $e_i(i) = \delta_{ij}$, δ_{ij} is the Kronecker delta function, therefore

$$\langle e_i, y\rangle = y(i) = 0,$$

which implies $y = 0$, therefore T is injective.

To prove T is surjective, let $f \in (\ell^p)^*$, if $f = 0$, then by taking $y = 0$, we have $f = f_0$. For $f \neq 0$, we define y_f as

$$y_f(i) = f(e_i).$$

We claim that $y_f \in \ell^q$ and $f = f_{y_f}$.

Indeed, for $p = 1$, $q = \infty$, for arbitrary $n \in \mathbb{N}$,

$$|y_f(n)| \leq |f(e_n)| \leq \|f\|_{(\ell^1)^*} \cdot \|e_n\|_1 = \|f\|_{(\ell^1)^*},$$

therefore $y_f \in \ell^\infty$with $\|y_f\|_{\ell^\infty} \leq \|f\|_{(\ell^1)^*}$. Moreover, by the definition, for arbitrary $x \in \ell^1$,

$$f(x) = f\left(\sum_{i=1}^{\infty} x(i)e_i\right) = \sum_{i=1}^{\infty} x(i)f(e_i) = \sum_{i=1}^{\infty} x(i)y_f(i) = \langle x, y_f \rangle,$$

which implies $f = f_{y_f}$, and

$$|f(x)| \leq |\langle x, y_f \rangle| \leq \|x\|_1 \|y_f\|_\infty,$$

implies $\|f\|_{(\ell^1)^*} \leq \|y_f\|_\infty$, therefore $\|y_f\|_\infty = \|f\|_{(\ell^1)^*}$.

For $1 < p, q < \infty$, and arbitrary $m \in \mathbb{N}$, we take

$$x_m(n) = \begin{cases} |y_f(n)|^{q-1} sgn(y_f) & , n \leq m, \\ 0 & , n > m, \end{cases}$$

then $x_m \in \ell^p$, indeed,

$$\|x_m\|_p = \left(\sum_{n=1}^{\infty} |x_m(n)|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{m} |y_f(n)|^q\right)^{\frac{1}{p}} < \infty.$$

Therefore on the one hand,

$$f(x_m) = \langle x_m, y_f \rangle = \sum_{n=1}^{m} x_m(n)y_f(n) = \sum_{n=1}^{m} |y_f(n)|^q,$$

on the other hand,

$$|f(x_m)| \leq \|f\|_{(\ell^p)^*} \cdot \|x_m\|_p \leq \|f\|_{(\ell^p)^*} \cdot \|y_f\|_q^{\frac{q}{p}},$$

therefore by letting m goes to infinity,

$$\|y_f\|_q^q \leq \|f\|_{(\ell^p)^*} \cdot \|y_f\|_q^{\frac{q}{p}},$$

which implies

$$\|y_f\|_q \leq \|f\|_{(\ell^p)^*},$$

therefore $y_f \in \ell^q$ with $\|y_f\|_q \leq \|f\|_{(\ell^p)^*}$. Moreover, by the definition, for arbitrary $x \in \ell^p$,

$$f(x) = f\left(\sum_{i=1}^{\infty} x(i)e_i\right) = \sum_{i=1}^{\infty} x(i)f(e_i) = \sum_{i=1}^{\infty} x(i)y_f(i) = \langle x, y_f \rangle,$$

which implies $f = f_{y_f}$, and

$$|f(x)| \leq |\langle x, y_f \rangle| \leq \|x\|_p \|y_f\|_q,$$

implies $\|f\|_{(\ell^p)^*} \leq \|y_f\|_q$, therefore $\|y_f\|_q = \|f\|_{(\ell^p)^*}$. □

Email address: jhzhang@math.cuhk.edu.hk